On Differential Equations Belonging to a Ternary Linearoid Group.

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It is the object of the present paper to investigate systems of differential equations which belong to a ternary linearoid group. The investigation is confined to those cases which are essentially distinct. The last paragraph is devoted to algebraic theorems on characteristic equations. Particular cases of these theorems were first noticed in attempting to treat two-parameter groups in their unreduced form. The results thus obtained have been generalized.

Differential equations belonging to linearoid groups have been studied by E. J. Wilczynski. He has proved the existence theorem* and obtained the differential equations belonging to a binary group.†

A group of linearoid transformations is defined by the system of equations

$$\eta_i = \phi_{i1}(x;\ a_1 \ldots . a_r)\, y_1 + \phi_{i2}(x;\ a_1 \ldots . a_r)\, y_2 + \ldots . \\ + \phi_{in}(x;\ a_1 \ldots . a_r)\, y_n, \quad (i = 1, 2 \ldots . n), \quad (1)$$

in which $\phi_{i\kappa}$ are uniform functions of x and of $a_1 \ldots . a_r$, and the r parameters a_κ are essential. The corresponding differential equations are such that if $y_1 \ldots y_n$ form a fundamental system of particular solutions, the general solutions are given by (1). Therefore, these solutions undergo substitutions contained in (1) when x makes circuits around the singular points of the differential equations. The study of such systems having *three* fundamental solutions is taken up in the

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* E. J. Wilczynski, "On Linearoid Differential Equations," American Journal of Mathematics, Vol. XXI, No. 4.

† E. J. Wilczynski, "On Continuous Binary Linearoid Groups and the Corresponding Differential Equations and Λ Functions," American Journal of Mathematics, Vol. XXII, No. 3.

present paper. The results obtained sufficiently indicate what is to be expected in general.

§1.—*One-Parameter Groups.*

The infinitesimal transformation of a one-parameter ternary linearoid group can be written

$$U(f) = (\psi_{11}y_1 + \psi_{12}y_2 + \psi_{13}y_3)q_1 + (\psi_{21}y_1 + \psi_{22}y_2 + \psi_{23}y_2)q_2 + (\psi_{31}y_1 + \psi_{32}y_2 + \psi_{33}y_3)q_3, \qquad \left(q_i = \frac{\partial f}{\partial q_i}\right), \tag{1}$$

where $\psi_{i\kappa}$ is a uniform function of x. The finite equations are obtained by integrating the linear system

$$\frac{\partial \eta_i}{\partial t} = \psi_{i1}\eta_1 + \psi_{i2}\eta_2 + \psi_{i3}\eta_3, \qquad (i = 1 \ldots . 3),$$

with the initial conditions $\eta_i = y_i$ for $t = 0$. The solutions are of the form

$$\eta_i = A_{i1}e^{\rho_1 t} + A_{i2}e^{\rho_2 t} + A_{i3}e^{\rho_3 t}, \qquad (i = 1 \ldots . 3),$$

where $\rho_1 \ldots . \rho_3$ are the roots, supposed distinct, of the cubic

$$\begin{vmatrix} \psi_{11} - \rho & \psi_{12} & \psi_{13} \\ \psi_{21} & \psi_{22} - \rho & \psi_{23} \\ \psi_{31} & \psi_{32} & \psi_{33} - \rho \end{vmatrix} = 0. \tag{2}$$

This equation will be called the *characteristic equation* of the infinitesimal transformation (1).

Determining $A_{i\kappa}$ from the systems

$$\left.\begin{aligned} (\psi_{11} - \rho_i)A_{1i} + \psi_{12}A_{2i} + \psi_{13}A_{3i} = 0, \\ \psi_{21}A_{1i} + (\psi_{22} - \rho_i)A_{2i} + \psi_{23}A_{3i} = 0, \\ \psi_{31}A_{1i} + \psi_{32}A_{2i} + (\psi_{33} - \rho_i)A_{3i} = 0, \end{aligned}\right\} \qquad (i = 1 \ldots . 3), \tag{3}$$

and putting $\eta_i = y_i$ for $t = 0$, the finite equations of the group become

$$\left.\begin{aligned}
\eta_1 = \frac{1}{\Delta}\Big[&[\lambda_1(\mu_2-\mu_3)e^{\rho_1 t} + \lambda_2(\mu_3-\mu_1)e^{\rho_2 t} + \lambda_3(\mu_1-\mu_2)e^{\rho_3 t}]\,y_1 \\
&+ [\lambda_1(\lambda_3-\lambda_2)e^{\rho_1 t} + \lambda_2(\lambda_1-\lambda_3)e^{\rho_2 t} + \lambda_3(\lambda_2-\lambda_1)e^{\rho_3 t}]\,y_2 \\
&+ [\lambda_1(\lambda_2\mu_3-\lambda_3\mu_2)e^{\rho_1 t} + \lambda_2(\lambda_3\mu_1-\lambda_1\mu_3)e^{\rho_2 t} + \lambda_3(\lambda_1\mu_2-\lambda_2\mu_1)e^{\rho_3 t}]\,y_3\Big], \\
\eta_2 = \frac{1}{\Delta}\Big[&[\mu_1(\mu_2-\mu_3)e^{\rho_1 t} + \mu_2(\mu_3-\mu_1)e^{\rho_2 t} + \mu_3(\mu_1-\mu_2)e^{\rho_3 t}]\,y_1 \\
&+ [\mu_1(\lambda_3-\lambda_2)e^{\rho_1 t} + \mu_2(\lambda_1-\lambda_3)e^{\rho_2 t} + \mu_3(\lambda_2-\lambda_1)e^{\rho_3 t}]\,y_2 \\
&+ [\mu_1(\lambda_2\mu_3-\lambda_3\mu_2)e^{\rho_1 t} + \mu_2(\lambda_3\mu_1-\lambda_1\mu_3)e^{\rho_2 t} + \mu_3(\lambda_1\mu_2-\lambda_2\mu_1)e^{\rho_3 t}]\,y_3\Big], \\
\eta_3 = \frac{1}{\Delta}\Big[&[(\mu_2-\mu_3)e^{\rho_1 t} + (\mu_3-\mu_1)e^{\rho_2 t} + (\mu_1-\mu_2)e^{\rho_3 t}]\,y_1 \\
&+ [(\lambda_3-\lambda_2)e^{\rho_1 t} + (\lambda_1-\lambda_3)e^{\rho_2 t} + (\lambda_2-\lambda_1)e^{\rho_3 t}]\,y_2 \\
&+ [(\lambda_2\mu_3-\lambda_3\mu_2)e^{\rho_1 t} + (\lambda_3\mu_1-\lambda_1\mu_3)e^{\rho_2 t} + (\lambda_1\mu_2-\lambda_2\mu_1)e^{\rho_3 t}]\,y_3\Big],
\end{aligned}\right\} \quad (4)$$

where $\lambda_i = \dfrac{A_{1i}}{A^i_3}$ and $\mu_i = \dfrac{A_{2i}}{A_{3i}}$ can be determined from (3), and where

$$\Delta = \begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ 1 & 1 & 1 \end{vmatrix}.$$

It is necessary and sufficient that the characteristic equation be reducible to a product of linear factors in order that (4) may generate a linearoid group, for then the coefficients of (4) are uniform functions of x.

Equations (4) no longer hold when Δ vanishes. This happens when the characteristic equation has a pair of equal roots. The question arises, can Δ vanish in any other case? This is best answered by considering (1) in its canonical form, namely:

$$U(f) = \phi_{11}y_1q_1 + (\phi_{21}y_1 + \phi_{22}y_2)q_2 + (\phi_{31}y_1 + \phi_{32}y_2 + \phi_{33}y_3)q_3,$$

a form to which it may always be reduced by a linearoid transformation. Such a transformation leaves unaltered the characteristic equation. Since it is apparent that ϕ_{11}, ϕ_{22} and ϕ_{33} are the roots of the characteristic equation corresponding to the canonical form, we have

$$\rho_i = \phi_{ii}, \qquad (i = 1,\ 2,\ 3).$$

Inserting the values of λ_i and μ_i, equations (3) become in this case

$$\left.\begin{aligned} (\phi_{11}-\rho_i)\,\lambda_i &= 0, \\ \phi_{21}\lambda_i + (\phi_{22}-\rho_i)\,\mu_i &= 0, \\ \phi_{31}\,\lambda_i + \phi_{32}\,\mu_i + \phi_{33} - \rho_i &= 0. \end{aligned}\right\} \tag{3a}$$

Supposing the roots distinct, (3a) shows that $\lambda_2 = \lambda_3 = 0$, and therefore

$$\Delta = \lambda_1\,(\mu_2 - \mu_3)\,.$$

Now λ_1 cannot vanish, for then μ_1 would vanish by (3a), which leads to the equality $\phi_{33} = \phi_{11}$, a result contrary to hypothesis. It can be shown by similar reasoning that $\mu_2 - \mu_3 \neq 0$. We therefore conclude that Δ vanishes only if the characteristic equation has at least one pair of equal roots.

The invariants of the group (4) are easily obtained. It is apparent from the form of the infinitesimal transformation that three relative linear invariants exist. Forming the expression $l\eta_1 + m\eta_2 + n\eta_3$ from (4) and applying the conditions for a relative invariant, we obtain

$$\begin{aligned} &(\mu_2-\mu_3)\,\eta_1 + (\lambda_3-\lambda_2)\,\eta_2 + (\lambda_2\mu_3 - \lambda_3\mu_2)\,\eta_3 \\ &\qquad\qquad = e^{\rho_1 t}\,[(\mu_2-\mu_3)\,y_1 + (\lambda_3-\lambda_2)\,y_2 + (\lambda_2\mu_3-\lambda_3\mu_2)\,y_3], \\ &(\mu_3-\mu_1)\,\eta_1 + (\lambda_1-\lambda_3)\,\eta_2 + (\lambda_3\mu_1 - \lambda_1\mu_3)\,\eta_3 \\ &\qquad\qquad = e^{\rho_2 t}\,[(\mu_3-\mu_1)\,y_1 + (\lambda_1-\lambda_3)\,y_2 + (\lambda_3\mu_1-\lambda_1\mu_3)\,y_3], \\ &(\mu_1-\mu_2)\,\eta_1 + (\lambda_2-\lambda_1)\,\eta_2 + (\lambda_1\mu_2 - \lambda_2\mu_1)\,\eta_3 \\ &\qquad\qquad = e^{\rho_3 t}\,[(\mu_1-\mu_2)\,y_1 + (\lambda_2-\lambda_1)\,y_2 + (\lambda_1\mu_2-\lambda_2\mu_1)\,y_3], \end{aligned}$$

which may be written in the form

$$H_1 = e^{\rho_1 t}\,Y_1, \quad H_2 = e^{\rho_2 t}\,Y_2, \quad H_3 = e^{\rho_3 t}\,Y_3. \tag{5}$$

From these relative invariants can be formed the two absolute invariants

$$\vartheta_1 = Y_1^{\rho_2}\,Y_2^{-\rho_1}, \quad \vartheta_2 = Y_1^{\rho_3}\,Y_3^{-\rho_1}.$$

The three differential invariants of the first order may be obtained from (5) They are

$$\frac{d}{dx}\left(\frac{1}{\rho_i}\log\,Y_i\right), \qquad (i = 1,\ 2,\ 3).$$

We therefore have the following system of differential equations belonging to (1),

$$\left.\begin{aligned} \frac{d}{dx}\log Y_i - \frac{1}{\rho_i}\log Y_i\frac{d\rho_i}{dx} &= f_i(x), \\ Y_1^{\rho_2}Y_2^{-\rho_1} = f_4(x), \quad Y_1^{\rho_3}Y_3^{-\rho_1} &= f_5(x), \end{aligned}\right\} \tag{6}$$

where, of course, the functions $f_i(x)$ are not independent but satisfy the relations

$$\left.\begin{aligned} \frac{1}{f_4}\;\frac{df_4}{dx} &= \frac{1}{\rho_1\rho_2}\;\frac{d(\rho_1\rho_2)}{dx}\log f_4 - \rho_2 f_1 - \rho_1 f_2, \\ \frac{1}{f_5}\;\frac{df_5}{dx} &= \frac{1}{\rho_1\rho_3}\;\frac{d(\rho_1\rho_3)}{dx}\log f_5 + \rho_3 f_1 - \rho_1 f_3. \end{aligned}\right\} \tag{6a}$$

The integration of (6) will introduce one arbitrary constant. The behavior of its solutions when the independent variable makes circuits around the singular points of the system will now be found.

Assume $f_i(x)$ $(i = 1 \ldots . 5)$ to be uniform functions of x. Equations (6) give on integration

$$\log Y_i = \frac{\rho_i}{\rho_i^0}\left[c_i + \rho_i^0\int_{x_0}\frac{f_i(x)}{\rho_i}\,dx\right], \qquad (i = 1,\ 2,\ 3), \tag{7}$$

where ρ_i^0 is the value of ρ_i for $x = x_0$. If $d_{i\kappa}$ denote the residual of $\dfrac{f_i(x)}{\rho_i}$ at the singular point a_κ $(\kappa = 1 \ldots . m)$, equations (7) show that $\log Y_i$ increases by $2\pi i\rho_i d_{i\kappa}$ when x makes a circuit around the singular point a_κ. Y_i therefore changes into

$$H_i = e^{2\pi i\rho_i d_{i\kappa}}Y_i. \tag{8}$$

The last two equations in (6) give, since f_4 and f_5 are uniform,

$$H_1^{\rho_2}H_2^{-\rho_1} = Y_1^{\rho_2}Y_2^{-\rho_1}, \text{ and } H_1^{\rho_3}H_3^{-\rho_1} = Y_1^{\rho_3}Y_3^{-\rho_1},$$

which leads to the conditions

$$e^{2\pi i\rho_1\rho_2(d_{1\kappa}-d_{2\kappa})} = 1\,; \quad e^{2\pi i\rho_1\rho_3(d_{1\kappa}-d_{3\kappa})} = 1\,;$$

therefore, if $\rho_1\rho_2$ and $\rho_1\rho_3$ are not constants,

$$d_{1\kappa} = d_{2\kappa} = d_{3\kappa},$$

i. e., $\dfrac{f_1}{\rho_1}$, $\dfrac{f_2}{\rho_2}$, $\dfrac{f_3}{\rho_3}$ *must have those singular points in common at which the residuals do not vanish, and the residuals at such common points must be equal.* Suppose on the contrary $\rho_1\rho_2$ to be a constant. The first equation in (6a) becomes

$$\frac{1}{f_4}\;\frac{df_4}{dx} = \rho_1\rho_2\left(\frac{f_1}{\rho_1} - \frac{f_2}{\rho_2}\right).$$

Introducing $f_4(x) = e^{F(x)}$, and requiring $F(x)$ to be a uniform function, we get, on integration,

$$F(x) = \rho_1 \rho_2 \int \left(\frac{f_1}{\rho_1} - \frac{f_2}{\rho_2} \right) dx,$$

from which it follows as before that $d_{1\kappa} = d_{2\kappa}$. Other special cases can be treated similarly. If ρ_i is zero, equations (6) break down. In this case the invariant system can be formed anew from (5). In all cases (8) can be made to agree with (5), provided we put $2\pi i\, d_{i\kappa} = t$.

The cases which arise when the characteristic equation has one pair of equal roots, or all of its roots equal, require separate treatment. The investigation is carried on most easily from the canonical form of the infinitesimal transformation. The various cases have been thus treated. No results essentially different from the above were obtained. In all cases there exists a system of functions y_1, y_2, y_3 with arbitrarily assigned branch-points a_κ, undergoing an arbitrarily assigned linearoid substitution A_κ contained in the one-parameter group, when x describes a closed path around a_κ.

§2.—*Two-Parameter Groups.*

Two-parameter groups may be treated in a variety of ways. Direct attack leads to some interesting relations which will be noticed in another place. The only practical solution is obtained by using Lie's types of linear groups. Nineteen such types of two-parameter linear homogeneous ternary groups exist (Lie. "Continuierliche Gruppen," p. 522). If functions of x are substituted for the constants appearing in these types, all types of *linearoid* groups will be obtained, For, suppose there were a two-parameter linearoid group which could not by a linearoid transformation be reduced to one of these linearoid types. Putting $x = a$, the group becomes linear. The result is a linear group which cannot by a linear transformation be reduced to a linear type, which is impossible. Treatment of one of these types will be sufficient. The following has been selected:

$$U_1 = y_3 q_2 + \mathrm{U}; \quad U_2 = y_3 q_1 + y_1 q_2 + \phi(x) \,.\, \mathrm{U}; \quad (U_1 U_2) = 0,$$

where we have put $\mathrm{U} = y_1 q_1 + y_2 q_2 + y_3 q_3$. The general infinitesimal transformation becomes

$$[(c_1 + c_2 \phi) y_1 + c_2 y_3]\, q_1 + [c_2 y_1 + (c_1 + c_2 \phi) y_2 + c_1 y_3]\, q_2 + [(c_1 + c_2 \phi) y_3]\, q_3. \quad (1)$$

The finite equations are obtained by integrating the system

$$\left.\begin{aligned} \frac{d\eta_1}{dt} &= (c_1 + c_2\phi)\,\eta_1 && + c_2\,\eta_3, \\ \frac{d\eta_2}{dt} &= c_2\,\eta_1 + (c_1 + c_2\phi)\,\eta_2 && + c_1\,\eta_3, \\ \frac{d\eta_3}{dt} &= && (c_1 + c_2\phi)\,\eta_3. \end{aligned}\right\} \tag{2}$$

The roots of the characteristic equation being equal, the solution has the form

$$\eta_i = e^{\rho t}(L_i + M_i t + N_i t^2), \qquad (i = 1, 2, 3).$$

Determination of the constants in the usual way gives, after putting $t = 1$, the system of equations

$$\left.\begin{aligned} \eta_1 &= e^{c_1 + c_2\phi}\,[y_1 + c_2 y_3], \\ \eta_2 &= e^{c_1 + c_2\phi}\,[c_2 y_1 + y_2 + (c_1 + \tfrac{1}{2}c_2^2)\,y_3], \\ \eta_3 &= e^{c_1 + c_2\phi}\,y_3. \end{aligned}\right\} \tag{3}$$

Putting in these equations $c_2 = 0$, we easily obtain the following invariants of the subgroup generated by U_1:

$$\left.\begin{aligned} &\vartheta_1 = \frac{y_1}{y_3}, \quad \vartheta_2 = y_1^{-1} e^{\frac{y_2}{y_3}}, \\ &\vartheta_3 = \frac{d}{dx}\log y_1, \quad \vartheta_4 = \frac{d}{dx}\log y_3, \quad \vartheta_5 = \frac{d}{dx}\,\frac{y_2}{y_3}. \end{aligned}\right\} \tag{4}$$

In order to obtain the invariants of group (3), it will be necessary to operate upon ϑ_i with U_2. Making use of the once extended operator

$$\begin{aligned} U_2'(f) = (\phi y_1 + y_3)\frac{\partial f}{\partial y_1} + (y_1 + \phi y_2)\frac{\partial f}{\partial y_2} + \phi y_3\frac{\partial f}{\partial y_3} \\ + (\phi y_1' + \phi' y_1 + y_3')\frac{\partial f}{\partial y_1'} + (y_1' + \phi y_2' + \phi' y_2)\frac{\partial f}{\partial y_2'} + (\phi y_3' + \phi' y_3)\frac{\partial f}{\partial y_3'}, \end{aligned}$$

we obtain after reducing

$$\left.\begin{aligned} &U_2'(\vartheta_1) = 1, \quad U_2'(\vartheta_2) = (\vartheta_1 + \vartheta_1^{-1} - \phi)\,\vartheta_2, \\ &U_2'(\vartheta_3) = \vartheta_1'\vartheta_1^{-2} + \phi', \quad U_2'(\vartheta_4) = \phi', \quad U_2'(\vartheta_5) = \vartheta_1'. \end{aligned}\right\} \tag{5}$$

An absolute invariant must be a function of ϑ_1 and ϑ_2. Denote it by $F(\vartheta_1 \vartheta_2)$. Applying the condition for invariance, we get

$$U_2(F) = U_2(\vartheta_1)\frac{\partial F}{\partial \vartheta_1} + U_2(\vartheta_2)\frac{\partial F}{\partial \vartheta_2} = 0.$$

Making use of (5), this becomes

$$\frac{\partial F}{\partial \vartheta_1} + (\vartheta_1 + \vartheta_1^{-1} - \phi)\,\vartheta_2\frac{\partial F}{\partial \vartheta_2} = 0,$$

the integral of which is found to be

$$F = \tfrac{1}{2}\vartheta_1 - \phi\vartheta_1 - \log(\vartheta_1 \vartheta_2).$$

The absolute invariant of (3) becomes therefore

$$\vartheta = \tfrac{1}{2}\left(\frac{y_1}{y_3}\right)^2 - \phi\frac{y_1}{y_3} - \frac{y_2}{y_3} + \log y_3.$$

Differential invariants are found by making use of equations (5). We obtain immediately the transformation group

$$\bar{\vartheta}_1 = \vartheta_1 + t, \quad \bar{\vartheta}_4 = \vartheta_4 + \phi' t, \quad \bar{\vartheta}_5 = \vartheta_5 + \vartheta_1' t.$$

The invariants of this one-parameter group are easily found. They are ϑ_1', $\phi'\vartheta_1 - \vartheta_4$, and $\vartheta_1'\vartheta_4 - \phi'\vartheta_5$, which are the required differential invariants of (3). The differential equations sought for are therefore

$$\left.\begin{aligned} \frac{d}{dx}\left(\frac{y_1}{y_3}\right) &= f_1(x), \\ \phi'\frac{y_1}{y_3} - \frac{d}{dx}\log y_3 &= f_2(x), \\ \phi'\frac{d}{dx}\left(\frac{y_2}{y_3}\right) - \frac{d}{dx}\left(\frac{y_1}{y_3}\right)\frac{d}{dx}\log y_3 &= f_3(x), \\ \tfrac{1}{2}\left(\frac{y_1}{y_3}\right)^2 - \phi\frac{y_1}{y_3} - \frac{y_2}{y_3} + \log y_3 &= f_4(x). \end{aligned}\right\} \quad (6)$$

This system is at once seen to be of the second order. The behavior of its solutions will not be investigated in detail, but is clear in general. They will be functions uniform everywhere except in the vicinity of certain singular points, and will undergo a linearoid substitution of the two-parameter group when x describes a closed path around one of these singular points. It can be shown

that the singular points and the corresponding substitutions may be chosen arbitrarily as in the case of a one-parameter group.

§3.—*Three-Parameter Groups.*

The only groups with which we shall be concerned in this and the following paragraphs will be non-integrable groups. All those occurring so far have been integrable. The results obtained may be taken as characteristic of such groups, the differential equations arising being simple combinations of linear differential equations.

A non-integrable three-parameter group can always be supposed to have the composition

$$(U_1 U_2) = U_1, \quad (U_1 U_3) = 2U_2, \quad (U_2 U_3) = U_3.$$

A non-integrable three-parameter linearoid group reduces to a non-integrable three-parameter linear group when x is put equal to a. Let U_1, U_2, U_3 generate a non-integrable three-parameter linearoid group. The substitution $x = a$ reduces it to a non-integrable group which is the linear transform of one or the other of the groups

$$\begin{array}{llll} 1. & V_1 = y_1 q_2, & V_2 = -\tfrac{1}{2} y_1 q_1 + \tfrac{1}{2} y_2 q_2, & V_3 = -y_2 q_1, \\ 2. & W_1 = 2y_2 q_1 + y_3 q_2, & W_2 = y_1 q_1 - y_3 q_3, & W_3 = -y_1 q_2 - 2y_2 q_3, \end{array}$$

since these are the only types of non-integrable ternary linear groups. U_1, U_2 and U_3 must therefore be the linearoid transform of either 1 or 2. These groups can therefore be considered instead of the linearoid group U_1, U_2, U_3.

Type **1.**—The finite equations of this group are known to be

$$\left.\begin{aligned} \eta_1 &= a y_1 + b y_2, \\ \eta_2 &= c y_1 + d y_2, \\ \eta_3 &= y_3, \end{aligned}\right\} \tag{1}$$

in which $ad - bc = 1$. There is one absolute invariant y_3. The differential invariants are

$$\vartheta_1 = y_1 y_2' - y_2 y_1', \quad \vartheta_2 = y_1 y_2'' - y_2 y_1'', \quad \vartheta_3 = y_1' y_2'' - y_2' y_1'.$$

In place of these, the system

$$\vartheta_1' = \frac{y_1 y_2'' - y_2 y_1''}{y_1 y_2' - y_2 y_1'}, \quad \vartheta_2' = \frac{y_1' y_2'' - y_2' y_1''}{y_1 y_2' - y_2 y_1'}$$

may be taken. Putting $\vartheta_1' = -f_1(x)$, $\vartheta_2' = f_2(x)$, and $y_3 = f_3(x)$, the system of invariant differential equations readily reduces to the following simple form:

$$\left.\begin{aligned} \frac{d^2 y_i}{dx^2} + f_1(x)\frac{dy_i}{dx} + f_2(x)\, y_i = 0, \\ y_3 - f_3(x) = 0, \end{aligned}\right\} \tag{2}$$
$$(i = 1, 2).$$

In this case, therefore, the linearoid system of invariant differential equations is merely the linearoid transform of a linear system. In order that its transformation group may be the special linear group, the further condition must be imposed that ϑ_1 be invariant under the transformations of this group.

Type 2.—The finite equations are

$$\left.\begin{aligned} \eta_1 &= a^2 y_1 + 2ab y_2 + b^2 y_3, \\ \eta_2 &= ac y_1 + (ad + bc)\, y_2 + bd y_3, \\ \eta_3 &= c^2 y_1 + 2cd y_2 + d^2 y_3, \end{aligned}\right\} \tag{3}$$

where $ad - bc = 1$. There is one absolute invariant $y_2^2 - y_1 y_3$. On account of (3), y_1, y_2 and y_3 must be solutions of a homogeneous linear differential equation of the third order. Our system becomes therefore

$$\left.\begin{aligned} \frac{d^3 y_i}{dx^3} + f_1(x)\frac{d^2 y_i}{dx^2} + f_2(x)\frac{dy_i}{dx} + f_3(x)\, y_i = 0 \quad, \\ y_2^2 - y_1 y_3 = f_4(x), \end{aligned}\right\} \tag{4}$$
$$(i = 1, 2, 3).$$

The corresponding linearoid system is the transform of (4) under the general linearoid substitution.

§4.—*Non-Integrable r-Parameter Groups whose Simple 3-Parameter Subgroup is an Invariant Subgroup.*

By a general theorem of Engel's,* every non-integrable group of continuous transformations contains a non-integrable 3-parameter subgroup. Suppose that this is an invariant subgroup. Then the general form of all non-integrable r-parameter ternary linearoid groups containing an invariant 3-parameter simple subgroup can be obtained. As shown by Wilczynski, we can take as the non-

* Lie, "Transformationsgruppen," Vol. III, p. 757.

integrable 3-parameter subgroup any *type* of non-integrable 3-parameter linear group. There are two cases therefore to be considered, corresponding to the two types of the last paragraph.

Case 1.—The invariant simple 3-parameter subgroup is

$$U_1 = 2y_2 q_1 + y_3 q_2, \quad U_2 = y_1 q_1 - y_3 q_3, \quad U_3 = -y_1 q_2 - 2y_2 q_3. \tag{1}$$

The remaining infinitesimal transformations have the form

$$U_\kappa = \sum_i^{(i=1\,\ldots\,3)} (\phi_{i1}^{(\kappa)} y_1 + \phi_{i2}^{(\kappa)} y_2 + \phi_{i3}^{(\kappa)} y_3)\, q_i.$$

Since the group (1) is an invariant subgroup, we have

$$(U_i U_\kappa) = c_{i\kappa 1} U_1 + c_{i\kappa 2} U_2 + c_{i\kappa 3} U_3, \qquad (i = 1, 2, 3\,;\ \varkappa = 4 \ldots r).$$

This becomes, making use of (1),

$$(U_i U_\kappa) = (c_{i\kappa 2} y_1 + 2c_{i\kappa 1} y_2)\, q_1 + (c_{i\kappa 3} y_1 - c_{i\kappa 1} y_3)\, q_2 - (2c_{i\kappa 3} y_2 + c_{i\kappa 2} y_3)\, q_3. \tag{4}$$

Equating coefficients in (4) with those obtained by direct calculation, gives the following equations of condition:

$$\left.\begin{array}{lll}
c_{12\kappa} = -2\phi_{21}^{(\kappa)} & 2c_{11\kappa} = 2(\phi_{11}^{(\kappa)} - \phi_{22}^{(\kappa)}) & 0 = \phi_{12}^{(\kappa)} - 2\phi_{23}^{(\kappa)} \\
c_{22\kappa} = 0 & 2c_{21\kappa} = -\phi_{12}^{(\kappa)} & 0 = -2\phi_{13}^{(\kappa)} \\
c_{32\kappa} = -\phi_{12}^{(\kappa)} & 2c_{31\kappa} = -\phi_{13}^{(\kappa)} & \\
c_{13\kappa} = -\phi_{31}^{(\kappa)} & 0 = 2\phi_{21}^{(\kappa)} - \phi_{32}^{(\kappa)} & c_{11\kappa} = \phi_{22}^{(\kappa)} - \phi_{33}^{(\kappa)} \\
c_{23\kappa} = \phi_{21}^{(\kappa)} & 0 = \phi_{12}^{(\kappa)} - 2\phi_{23}^{(\kappa)} & c_{21\kappa} = -\phi_{23}^{(\kappa)} \\
c_{33\kappa} = \phi_{11}^{(\kappa)} - \phi_{55}^{(\kappa)} & & c_{31\kappa} = \phi_{13}^{(\kappa)} \\
0 = \phi_{31}^{(\kappa)} & -2c_{13\kappa} = 2\phi_{31}^{(\kappa)} & -2c_{12\kappa} = \phi_{32}^{(\kappa)} \\
0 = 2\phi_{21}^{(\kappa)} - \phi_{32}^{(\kappa)} & -2c_{23\kappa} = \phi_{32}^{(\kappa)} & -2c_{22\kappa} = 0 \\
 & -2c_{33\kappa} = 2(\phi_{22}^{(\kappa)} - \phi_{33}^{(\kappa)}) & -2c_{32\kappa} = 2\phi_{23}^{(\kappa)}
\end{array}\right\} \tag{5}$$

From this system we deduce

$$\phi_{13}^{(\kappa)} = \phi_{21}^{(\kappa)} = \phi_{31}^{(\kappa)} = \phi_{32}^{(\kappa)} = 0,$$
$$\phi_{11}^{(\kappa)} - \phi_{22}^{(\kappa)} = \phi_{22}^{(\kappa)} - \phi_{33}^{(\kappa)} = c_{11\kappa}, \quad \phi_{12}^{(\kappa)} - 2\phi_{23}^{(\kappa)} = 0, \quad \phi_{23}^{(\kappa)} = -c_{21\kappa}.$$

U_κ therefore assumes the form

$$U_\kappa = \phi_{11}^{(\kappa)} y_1 q_1 + (\phi_{11}^{(\kappa)} - c_{11\kappa})\, y_2 q_2 + (\phi_{11}^{(\kappa)} - 2c_{11\kappa})\, y_3 q_3 - c_{21\kappa}(2y_2 q_1 + y_3 q_2).$$

Omitting terms of the form $c_1 U_1 + c_2 U_2 + c_3 U_3$, this becomes

$$U_\kappa = \phi_\kappa (y_1 q_1 + y_2 q_2 + y_3 q_3), \qquad (\varkappa = 4 \ldots r). \tag{6}$$

The finite equations of this group are found to be

$$\left.\begin{aligned} \eta_1 &= e^{\Phi}(a^2 y_1 + 2ab y_2 + b^2 y_3), \\ \eta_2 &= e^{\Phi}(ac y_1 + (ad+bc) y_2 + bd y_3), \\ \eta_3 &= e^{\Phi}(c^2 y_1 + 2cd y_2 + d^2 y_3), \end{aligned}\right\} \tag{7}$$

where we have put $\Phi = c_4 \phi_4 + \dots + c_r \phi_r$ and $ab - bc = 1$. Group (7) possesses the relative invariant $y_2^2 - y_1 y_3$, the transformation equation being

$$\eta_2^2 - \eta_1 \eta_3 = e^{2\Phi}(y_2^2 - y_1 y_3). \tag{7a}$$

Differential invariants are obtained as follows: The minors of y''', y'', y' and y in the determinant

$$\begin{vmatrix} y''' & y_1''' & y_2''' & y_3''' \\ y'' & y_1'' & y_2'' & y_3'' \\ y' & y_1' & y_2' & y_3' \\ y & y_1 & y_2 & y_3 \end{vmatrix}$$

are invariants of U_1, U_2 and U_3. Denote them by p, q, r and s respectively. An attempt will now be made to find functions of p, q, r and s which permit all the transformations of (7). Forming with this in view the three times extended operator

$$U_\kappa'''(f) = \sum_i^{1\dots 3}\Big[\phi_\kappa y_i \frac{\partial f}{\partial y_i} + (\phi_\kappa y_i' + \phi_\kappa' y_i)\frac{\partial f}{\partial y_i'} + (\phi_\kappa y_i'' + 2\phi_\kappa' y_i' + \phi_\kappa'' y_i)\frac{\partial f}{\partial y_i''} + (\phi_\kappa y_i''' + 3\phi_\kappa' y_i'' + 3\phi_\kappa' y_i' + \phi_\kappa''' y_i)\frac{\partial f}{\partial y_i'''}\Big],$$

and operating upon p, q, r and s, we finally obtain

$$\left.\begin{aligned} U_\kappa(s) &= 3\phi_\kappa s, \\ U_\kappa(p) &= 3\phi_\kappa' s + 3\phi_\kappa p, \\ U_\kappa(q) &= -3\phi_\kappa' s + 2\phi_\kappa' p + 3\phi_\kappa q, \\ U_\kappa(r) &= \phi_\kappa''' s - \phi_\kappa'' p + \phi_\kappa' q + 3\phi_\kappa r. \end{aligned}\right\} \tag{8}$$

Integration of (8) gives the following quaternary $(r-3)$-parameter linearoid group to which p, q, r and s are subject under the transformations of the group (7):

$$\left.\begin{aligned} \bar{s} &= e^{3\Phi} s, \\ \bar{p} &= e^{3\Phi}[3\Phi' s + p], \\ \bar{q} &= e^{3\Phi}[3(\Phi'^2 - \Phi'') s + 3\Phi' p + q], \\ \bar{r} &= e^{3\Phi}[(\Phi'^3 - 3\Phi'\Phi'' + \Phi''') s + (\Phi'^2 - \Phi'') p + \Phi' q + r]. \end{aligned}\right\} \tag{9}$$

From these can be obtained the relations

$$\left.\begin{aligned}
\left(\frac{\bar{p}}{s}\right) &= \frac{p}{s} + 3\Phi', \\
\left(\frac{\bar{q}}{s}\right) &= \frac{q}{s} + 2\Phi'\frac{p}{s} + 3(\Phi'^2 - \Phi''), \\
\left(\frac{\bar{r}}{s}\right) &= \frac{r}{s} + \Phi'\frac{q}{s} + (\Phi'^2 - \Phi'')\frac{q}{s} + \Phi'^3 + \Phi''' - 3\Phi'\Phi'',
\end{aligned}\right\} \qquad (10)$$

and from (7a),

$$\log(\eta_2^2 - \eta_1\eta_3) = \log(y_2^2 - y_1 y_3) + 2\Phi. \qquad (10a)$$

From (10) we form the invariants

$$\left.\begin{aligned}
\vartheta_1 &= \frac{q}{s} - \tfrac{1}{3}\left(\frac{p}{s}\right)^2 + \frac{d}{dx}\left(\frac{p}{s}\right), \\
\vartheta_2 &= -\tfrac{2}{27}\left(\frac{p}{s}\right)^3 + \tfrac{1}{3}\frac{p}{s}\cdot\frac{q}{s} - \frac{r}{s} + \tfrac{1}{3}\frac{d^2}{dx^2}\left(\frac{p}{s}\right).
\end{aligned}\right\} \qquad (11)$$

The first equation in (10) requires that $\frac{p}{s}$ satisfy a non-homogeneous linear differential equation of order $r - 3$, the corresponding homogeneous equation having $\phi_4' \ldots . \phi_r'$ as the members of a fundamental system. The differential equations sought for become

$$\left.\begin{aligned}
\frac{d^{r-3}}{dx^{r-3}}\left(\frac{p}{s}\right) + g_1(x)\frac{d^{r-4}}{dx^{r-4}}\left(\frac{p}{s}\right) + \ldots . \\
+ g_{r-4}(x)\frac{d}{dx}\left(\frac{p}{s}\right) + g_{r-3}(x)\frac{p}{s} &= f_1(x), \\
\tfrac{1}{3}\frac{d^2}{dx^2}\left(\frac{p}{s}\right) - \tfrac{2}{27}\left(\frac{p}{s}\right)^3 + \tfrac{1}{3}\frac{p}{s}\cdot\frac{q}{s} - \frac{r}{s} &= f_2(x), \\
\frac{d}{dx}\left(\frac{p}{s}\right) - \tfrac{1}{3}\left(\frac{p}{s}\right)^2 + \frac{q}{s} &= f_3(x).
\end{aligned}\right\} \qquad (12)$$

This system determines $\frac{p}{s}$, $\frac{q}{s}$ and $\frac{r}{s}$. We now determine y_1, y_2 and y_3 from the equation

$$\frac{d^3y}{dx^3} - \frac{p}{s}\frac{d^2y}{dx^2} + \frac{q}{s}\frac{dy}{dx} - \frac{r}{s}y = 0, \qquad (13)$$

of which they are fundamental solutions. The additional equation

$$\frac{d}{dx}\log(y_2^2-y_1y_3)-\tfrac{2}{3}\frac{p}{s}=f_4(x), \tag{13a}$$

obtained from (10) and (10a), must also be satisfied, which condition reduces the group of equation (13) to a 3-parameter group.

It now remains to verify that the solutions of (16) belong to the r-parameter group (7), and that the constants available can be chosen so as to make these solutions undergo arbitrary substitutions of (7) when the independent variable x makes circuits around a finite number of arbitrarily assigned singular points. Let $\phi_1' \ldots . \phi_{r-3}'$ be a fundamental system of solutions of the homogeneous equation corresponding to

$$\frac{d^{r-3}}{dx^{r-3}}\left(\frac{p}{s}\right)+g_1(x)\frac{d^{r-4}}{dx^{r-4}}\left(\frac{p}{s}\right)+\ldots . +g_{r-3}(x)\,.\,\frac{p}{s}=f_1(x).$$

According to the theory of linear differential equations, its general solution can be written

$$\left(\frac{\bar{p}}{s}\right)=\frac{p}{s}+\sum_{\kappa}^{3\ldots . r}c_\kappa\phi_\kappa'=\frac{p}{s}+3\Phi', \tag{14}$$

where $\frac{p}{s}$ is a particular solution. If x makes circuits around arbitrarily assigned branch-points corresponding to the singular points of $f_1(x)$, supposed uniform these solutions will undergo substitutions belonging to (14). From the third equation in (12) we obtain

$$\left(\frac{q}{s}\right)=f_3(x)+\tfrac{1}{3}\left(\frac{p}{s}\right)^2-\frac{d}{dx}\left(\frac{p}{s}\right);$$

making use of (14), the substitution for $\frac{q}{s}$ becomes, supposing $f_3(x)$ uniform

$$\left(\frac{\bar{q}}{s}\right)=f_3(x)+\tfrac{1}{3}\left(\frac{\bar{p}}{s}\right)-\frac{d}{dx}\left(\frac{\bar{p}}{s}\right)=\frac{q}{s}+2\Phi'.\frac{p}{s}+3(\Phi'^2-\Phi''). \tag{15}$$

Substituting (14) and (15) in the second equation of (12), supposing $f_2(x)$ to be uniform, we obtain finally

$$\left(\frac{\bar{r}}{s}\right)=\left(\frac{r}{s}\right)+(\Phi'^2-\Phi'')\frac{p}{s}+\Phi'\frac{q}{s}+\Phi'^3+\Phi'''-3\Phi'\Phi''. \tag{16}$$

It is clear, therefore, that the solutions of (12) undergo substitutions of the group (9) when x makes circuits around the branch-points of the system. It remains to show that the solutions of (13) in that case undergo substitutions contained in (7). Equation (13) becomes, after making the transformation $y = e^{\frac{1}{3}\int \frac{p}{s} dx} \eta$,

$$\frac{d^3\eta}{dx^3} + \vartheta_1 \frac{d\eta}{dx} + \vartheta_2 \eta = 0. \tag{17}$$

ϑ_i is given by (11). The coefficients being invariant, the solutions of (17) are subject to linear substitutions with constant coefficients,

$$\eta_i = \lambda_{i1}\eta_1 + \lambda_{i2}\eta_2 + \lambda_{i3}\eta_3, \qquad (i = 1, 2, 3), \tag{18}$$

and, therefore, after a circuit around the branch-points,

$$\bar{y}_i = e^{\frac{1}{3}\int \left(\frac{\bar{p}}{q}\right) dx} \bar{\eta}_i = e^{\frac{1}{3}\int \frac{p}{q} dx + \Phi} \left[\lambda_{i1}\eta_1 + \lambda_{i2}\eta_2 + \lambda_{i3}\eta_3\right],$$

which becomes, after change of variable,

$$\bar{y}_i = e^{\Phi} \left[\lambda_{i1} y_1 + \lambda_{i2} y_2 + \lambda_{i3} y_3\right]. \tag{19}$$

The λ's in this system are not independent. For, since y_1, y_2 and y_3 must satisfy (13a), the transformation (18) must be such as to leave $y_2^2 - y_1 y_3$ invariant. This additional condition reduces the nine-parameter group (18) to the three-parameter subgroup (7). The verification is therefore complete. For to prove that the substitutions of the subgroup (7) may be arbitrarily assigned is a problem in the theory of linear differential equations.

The second type will now be discussed. The non-integrable subgroup ${}_3G$ for this case is

$$U_1 = y_1 q_1, \quad U_2 = \tfrac{1}{2}(-y_1 q_1 + y_2 q_2), \quad U_3 = -y_2 q_1. \tag{20}$$

By a process similar to that used in the case of the first type, the following general form for this group was obtained:

$$U_\kappa = \phi_\kappa(x)(y_1 q_1 + y_2 q_2) + \psi_\kappa(x) y_3 q_3 + a_\kappa y_2 q_2, \qquad (\kappa = 4 \ldots . r). \tag{21}$$

Forming the general transformation and putting

$$\Sigma c_\kappa \phi_\kappa = \Phi; \quad \Sigma c_\kappa \psi_\kappa = \Psi; \quad \Sigma c_\kappa a_\kappa = a,$$

we obtain finally the finite equations of the group

$$\left.\begin{aligned}\eta_1 &= e^{\Phi}\left[\frac{1}{\lambda_2-\lambda_1}(\lambda_2 e^{\rho_1}-\lambda_1 e^{\rho_2})\,y_1+\frac{1}{\lambda_1-\lambda_2}(e^{\rho_1}-e^{\rho_2})\,y_2\right],\\ \eta_2 &= e^{\Phi}\left[\frac{\lambda_1\lambda_2}{\lambda_1-\lambda_2}(e^{\rho_1}-e^{\rho_2})\,y_1+\frac{1}{\lambda_2-\lambda_1}(-\lambda_1 e^{\rho_1}+\lambda_2 e^{\rho_2})\,y_2\right],\\ \eta_3 &= e^{\Psi}\,y_3\end{aligned}\right\}\tag{22}$$

where ρ_i are roots of the equation

$$\rho^2 - a\rho - \tfrac{1}{2}ac_2 - \tfrac{1}{2}c_2^2 + c_1c_3 = 0,$$

and where λ_i is given by

$$\lambda_i^1 = -\frac{c_2}{2c_3} - \frac{\rho_i}{c_3}.$$

The invariants of G_3 can be taken as

$$u = y_1y_2' - y_2y_1',\quad v = y_1y_2'' - y_2y_1'',\quad w = y_1'y_2'' - y_2'y_1'',\quad s = y_3. \tag{23}$$

Operating upon these with the twice-extended operator $U_\kappa''(f)$, we obtain

$$\left.\begin{aligned}&U_\kappa(u) = (2\phi_\kappa + a_\kappa)\,u, && U_\kappa(w) = -\phi''u + \phi'v + (2\phi_\kappa + a_\kappa)\,w,\\ &U_\kappa(v) = 2\phi'u + (2\phi_\kappa + a_\kappa)\,v, && U_\kappa(s) = \psi_\kappa s,\end{aligned}\right\}\tag{24}$$

the integration of which gives the group induced by (22) on u, v, w and s,

$$\left.\begin{aligned}&\bar{u} = e^{2\Phi+a}.u && \bar{w} = e^{2\Phi+a}\left[(\Phi'^2-\Phi'')\,u + \Phi'v + w\right],\\ &\bar{v} = e^{2\Phi+a}\left[2\phi'u + v\right], && \bar{s} = e^{\Psi}.s,\end{aligned}\right\}\tag{25}$$

and, therefore,

$$\left(\frac{\bar{v}}{u}\right) = \frac{v}{u} + 2\Phi',\quad \log\bar{s} = \log s + \Psi, \tag{26}$$

leading to the absolute invariant

$$\vartheta_1 = \tfrac{1}{2}\frac{d}{dx}\left(\frac{v}{u}\right) - \tfrac{1}{4}\left(\frac{v}{u}\right)^2 + \frac{w}{u}. \tag{27}$$

According to (26), $\dfrac{v}{u}$ satisfies a non-homogeneous linear differential equation of order $r-3$. The system of differential equations belonging to (22) becomes therefore

$$\left.\begin{aligned}\frac{d^{r-3}}{dx^{r-3}}\left(\frac{v}{u}\right) + g_1(x)\frac{d^{r-4}}{dx^{r-4}}\left(\frac{v}{u}\right) + \dots + g_{r-3}(x)\left(\frac{v}{u}\right) &= f_1(x),\\ \tfrac{1}{2}\frac{d}{dx}\left(\frac{v}{u}\right) - \tfrac{1}{4}\left(\frac{v}{u}\right)^2 + \frac{w}{u} &= f_2(x).\end{aligned}\right\}\tag{28}$$

Since $\log y_3$ undergoes the same substitutions as $\frac{v}{u}$, it is determined by an equation of the form

$$\log y_3 = h_0(x) + h_1(x)\frac{v}{u} + h_2(x)\frac{d}{dx}\,\frac{v}{u} + \ldots + h_{r-3}(x)\frac{d^{r-4}}{dx^{r-4}}\left(\frac{v}{u}\right), \quad (29)$$

in which $h_1 \ldots h_{r-3}$ are uniform functions, determined by the system of equations

$$h_1\phi_i^{(1)} + h_2\phi_i^{(2)} + \ldots + h_{r-3}\phi_i^{r-3} = \psi_i, \qquad (i = 4 \ldots r), \quad (29a)$$

and where h_0 is an arbitrary uniform function. $\frac{v}{u}$ and $\frac{w}{u}$ having been found from (28), y_1 and y_2 are determined from

$$\frac{d^2y_i}{dx^2} - \frac{v}{u}\,\frac{dy_i}{dx} + \frac{w}{u}\,y_i = 0, \qquad (i = 1,\ 2) \quad (30)$$

The behavior of the functions y_1, y_2 and y_3, as defined by (28), (29) and (30), is obtained as in the first case. The results are essentially the same, and will therefore not be given.

§5.—*Non-Integrable r-Parameter Groups whose Simple Three-Parameter Subgroup is not an Invariant Subgroup.*

4-*Parameter Groups.*—There are no 4-parameter groups of this class. For we have (Lie, Tr. Gr., III, p. 723) for all non-integrable 4-parameter groups the one typical composition $(U_i U_4) = 0$, where $U_i\,(i = 1, 2, 3)$ are the infinitesimal transformations of the simple 3-parameter subgroup.

5-*Parameter Groups.*—All groups of this kind have the same composition as (Tr. Gr., III, p. 736)

$$U_1 = xq, \quad U_2 = xp - yq, \quad U_3 = yp, \quad U_4 = p, \quad U_5 = q. \quad (1)$$

Assuming G_3 to be of the first form or

$$U_1 = y_1 q_2, \quad U_2 = y_1 q_1 - y_2 q_2, \quad U_3 = y_2 q_1,$$

we obtain the following linearoid types, having the same composition as (1):

$$\left.\begin{array}{lll} \text{A.} & G_3, & U_4 = \phi(x)\, y_3 q_1, \quad U_5 = \phi(x)\, y_3 q_2. \\ \text{B.} & G_3, & U_4 = \phi(x)\, y_2 q_3, \quad U_5 = \phi(x)\, y_1 q_3. \end{array}\right\} \quad (2)$$

The finite equations of group A are, as may be easily verified,

$$\left.\begin{aligned}\eta_1 &= ay_1 + by_2 + e\phi y_3,\\ \eta_2 &= cy_1 + dy_2 + f\phi y_3,\\ \eta_3 &= y_3,\end{aligned}\right\} \tag{3}$$

where a, b, c, d, e, f are constants subject to the condition

$$ad - bc = 1,$$

which, moreover, we may suppress, thus obtaining a 6-parameter group. Put

$$\frac{1}{\phi}\,\frac{y_1}{y_3} = Y_1, \quad \frac{1}{\phi}\,\frac{y_2}{y_3} = Y_2. \tag{4}$$

Then Y_1 and Y_2 are transformed by the general linear group

$$\left.\begin{aligned}H_1 &= aY_1 + bY_2 + e,\\ H_2 &= cY_1 + dY_2 + f,\end{aligned}\right\} \tag{5}$$

so that Y_1', Y_2' are transformed by the general linear homogeneous group. Y_1', Y_2' therefore constitute a fundamental system of solutions of a linear homogeneous differential equation of the second order, and Y_1, Y_2 themselves are integrals of such functions, while y_3 is itself a uniform function of x.

The finite equation of group B may be written :

$$\left.\begin{aligned}\eta_1 &= ay_1 + by_2,\\ \eta_2 &= cy_1 + dy_2,\\ \eta_3 &= e\phi y_1 + f\phi y_2 + y_3.\end{aligned}\right\} \tag{6}$$

Therefore, y_1, y_2 form a fundamental system of a linear differential equation of the second order, say

$$y'' + py' + qy = 0, \tag{7}$$

while $\frac{1}{\phi}\,y_3$ is a solution of the non-homogeneous linear differential equation

$$y'' + py' + qy = r, \tag{8}$$

whose left member is identical with the left member of (7). For if Y denotes any solution of (8), its general solution is

$$H = Y + ey_1 + fy_2,$$

where e and f are two arbitrary constants.

If G_3 is assumed to be of the second form

$$U_1 = 2y_2 q_1 + y_3 q_2, \quad U_2 = -2y_1 q_1 + 2y_3 q_3, \quad U_3 = y_1 q_2 + 2y_2 q_3,$$

it will be found that there are no 5-parameter groups of this class. By making use of Lie's types of 6-parameter non-integrable groups, the same was found to be true of 6-parameter groups also.

The results obtained for 5-parameter groups of this class sufficiently indicate what is to be expected in general. The study of r-parameter groups belonging to this class will therefore not be followed out any further.

From the preceding investigation, it appears that finite linearoid groups do not succeed in defining any essentially new functions. It has been shown by Wilczynski* that higher transcendental functions exist, whose multiformity is qualitatively of the same kind as that of the functions occurring in this paper Their group, however, is not contained in any finite continuous linearoid group. It is very doubtful if they satisfy any algebraic differential equations.

§6.—*On the Characteristic Equation belonging to Certain Linear and Linearoid Groups.*

The following theorems about characteristic equations, although most easily stated in terms of group theory, are of a purely algebraic nature. Consider a two-parameter group with the composition

$$(U_1 U_2) = a U_1, \tag{1}$$

where $a \neq 0$, and may, moreover, without loss of generality, be taken equal to unity. Writing

$$\left.\begin{aligned} U_1 &= (\phi_{11} y_1 + \phi_{12} y_2) q_1 + (\phi_{21} y_1 + \phi_{22} y_2) q_2, \\ U_2 &= (\psi_{11} y_1 + \psi_{12} y_2) q_1 + (\psi_{21} y_1 + \psi_{22} y_2) q_2, \end{aligned}\right\} \tag{2}$$

we derive from (1)

$$\left.\begin{aligned} A_1 &\equiv \phi_{21}\psi_{12} - \phi_{12}\psi_{21} &&= \phi_{11}, \\ A_2 &\equiv \phi_{12}\psi_{11} - \phi_{11}\psi_{12} + \phi_{22}\psi_{12} - \phi_{12}\psi_{22} &&= \phi_{12}, \\ A_3 &\equiv \phi_{11}\psi_{21} - \phi_{21}\psi_{11} + \phi_{21}\psi_{22} - \phi_{22}\psi_{21} &&= \phi_{21}, \\ A_4 &\equiv \phi_{12}\psi_{21} - \phi_{21}\psi_{12} &&= \phi_{22}, \end{aligned}\right\} \tag{3}$$

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* American Journal of Mathematics, Vol. XXI, No. 3.

where A_i stands for the left members of these equations. The following identities are easily verified:

$$\left.\begin{aligned} A_1 + A_4 &= 0, \\ \phi_{21} A_2 + \phi_{12} A_3 + (\phi_{11} - \phi_{22}) A_1 &= 0, \\ \psi_{21} A_2 + \psi_{12} A_3 + (\psi_{11} - \psi_{22}) A_1 &= 0, \end{aligned}\right\} \tag{4}$$

from which we derive

$$\phi_{11} + \phi_{22} = 0, \quad (\phi_{11} - \phi_{22})\, \phi_{11} + 2\phi_{12}\, \phi_{21} = 0.$$

The coefficients of the characteristic equation belonging to U_1,

$$\rho^2 - (\phi_{11} + \phi_{22})\, \rho + \phi_{11}\, \phi_{22} - \phi_{12}\, \phi_{21} = 0$$

are therefore zero. Both of its roots are therefore zero.*

In the ternary case the system corresponding to (3) becomes

$$\left.\begin{aligned}
A_1 &\equiv \phi_{21}\psi_{12} - \phi_{12}\psi_{21} + \phi_{31}\psi_{13} - \phi_{13}\psi_{31} &&= \phi_{11}, \\
A_2 &\equiv \phi_{12}\psi_{11} - \phi_{11}\psi_{12} + \phi_{22}\psi_{12} - \phi_{12}\psi_{22} + \phi_{32}\psi_{13} - \phi_{13}\psi_{32} &&= \phi_{12}, \\
A_3 &\equiv \phi_{13}\psi_{11} - \phi_{11}\psi_{13} + \phi_{23}\psi_{12} - \phi_{12}\psi_{23} + \phi_{33}\psi_{13} - \phi_{13}\psi_{33} &&= \phi_{13}, \\
A_4 &\equiv \phi_{11}\psi_{21} - \phi_{21}\psi_{11} + \phi_{21}\psi_{22} - \phi_{22}\psi_{21} + \phi_{31}\psi_{23} - \phi_{23}\psi_{31} &&= \phi_{21}, \\
A_5 &\equiv \phi_{12}\psi_{21} - \phi_{21}\psi_{12} + \phi_{32}\psi_{23} - \phi_{23}\psi_{32} &&= \phi_{22}, \\
A_6 &\equiv \phi_{13}\psi_{21} - \phi_{21}\psi_{13} + \phi_{23}\psi_{22} - \phi_{22}\psi_{23} + \phi_{33}\psi_{23} - \phi_{23}\psi_{33} &&= \phi_{23}, \\
A_7 &\equiv \phi_{11}\psi_{31} - \phi_{31}\psi_{11} + \phi_{21}\psi_{32} - \phi_{32}\psi_{21} + \phi_{31}\psi_{33} - \phi_{33}\psi_{31} &&= \phi_{31}, \\
A_8 &\equiv \phi_{12}\psi_{31} - \phi_{31}\psi_{12} + \phi_{22}\psi_{32} - \phi_{32}\psi_{22} + \phi_{32}\psi_{33} - \phi_{33}\psi_{32} &&= \phi_{32}, \\
A_9 &\equiv \phi_{13}\psi_{31} - \phi_{31}\psi_{13} + \phi_{22}\psi_{32} - \phi_{32}\psi_{22} &&= \phi_{33},
\end{aligned}\right\} \tag{5}$$

from which we obtain the identities

$$\left.\begin{aligned}
& A_1 + A_5 + A_9 \equiv 0, \\
& \phi_{21} A_2 + \phi_{12} A_4 + \phi_{31} A_3 + \phi_{13} A_7 + \phi_{32} A_6 + \phi_{23} A_8 \\
& \qquad + (\phi_{11} - \phi_{33}) A_1 + (\phi_{22} - \phi_{33}) A_5 \equiv 0, \\
& (\phi_{22}\phi_{33} - \phi_{23}\phi_{32}) A_1 + (\phi_{23}\phi_{31} - \phi_{21}\phi_{33}) A_2 + (\phi_{21}\phi_{32} - \phi_{22}\phi_{31}) A_3 \\
& \qquad + (\phi_{13}\phi_{32} - \phi_{12}\phi_{33}) A_4 + (\phi_{11}\phi_{33} - \phi_{13}\phi_{31}) A_5 + (\phi_{12}\phi_{31} - \phi_{11}\phi_{32}) A_6 \\
& \qquad + (\phi_{12}\phi_{23} - \phi_{22}\phi_{13}) A_7 + (\phi_{13}\phi_{21} - \phi_{11}\phi_{23}) A_8 + (\phi_{11}\phi_{22} - \phi_{12}\phi_{21}) A_9 \equiv 0,
\end{aligned}\right\} \tag{6}$$

and two others, differing from these only in having ψ in place of ϕ. Making use of (5), we find

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* This result was derived by Wilczynski (American Journal, Vol. XXII, No. 3, p. 208), who also noticed there that the roots of the characteristic equation of $U_2(f)$ differ by unity.

$$\left.\begin{aligned}\phi_{11}+\phi_{22}+\phi_{33}=0\,,\\(\phi_{11}-\phi_{33})\phi_{11}+(\phi_{22}-\phi_{33})\phi_{22}+2(\phi_{12}\phi_{21}+\phi_{13}\phi_{31}+\phi_{23}\phi_{32})=0\,,\\\phi_{11}\phi_{22}\phi_{33}+\phi_{21}\phi_{32}\phi_{13}+\phi_{31}\phi_{23}\phi_{12}-\phi_{13}\phi_{22}\phi_{31}-\phi_{23}\phi_{32}\phi_{11}-\phi_{12}\phi_{21}\phi_{33}=0\,.\end{aligned}\right\}\quad(7)$$

Combining the first equation with the second, the left members, it will be noticed, agree with the coefficients of the characteristic equation belonging to U_1, namely,

$$\rho^3-(\phi_{11}+\phi_{22}+\phi_{33})\rho^2+(\phi_{11}\phi_{33}+\phi_{22}\phi_{33}+\phi_{11}\phi_{22}-\phi_{13}\phi_{31}-\phi_{32}\phi_{23}-\phi_{12}\phi_{21})\rho$$
$$-\phi_{11}\phi_{22}\phi_{33}+\phi_{21}\phi_{32}\phi_{13}+\phi_{31}\phi_{23}\phi_{12}-\phi_{13}\phi_{22}\phi_{31}-\phi_{23}\phi_{32}\phi_{11}-\phi_{12}\phi_{21}\phi_{33}=0,$$

from which the theorem follows for this case also. The general case will now be considered.

A group of composition (1) is integrable. U_1 and U_2 can therefore be transformed simultaneously by a linearoid transformation into the canonical form

$$\left.\begin{aligned}U_1=\phi_{11}y_1q_1+(\phi_{21}y_1+\phi_{22}y_2)q_2+(\phi_{31}y_1+\phi_{32}y_2+\phi_{33}y_3)q_3+\cdots\\U_2=\psi_{11}y_1q_1+(\psi_{21}y_1+\psi_{22}y_2)q_2+(\psi_{31}y_1+\psi_{32}y_2+\psi_{33}y_3)q_3+\cdots\end{aligned}\right\}\quad(8)$$

Such a transformation does not alter the characteristic equation or the composition of the group.

Making use of equation (1), we obtain

$$\begin{aligned}&(\psi_{11}q_1+\psi_{21}q_2+\cdots+\psi_{n1}q_n)\phi_{11}y_1\\&+(\psi_{22}q_2+\psi_{32}q_3+\cdots+\psi_{n2}q_n)(\phi_{21}y_1+\phi_{22}y_2)\\&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\&+(\psi_{n-1,n-1}q_{n-1}+\psi_{n,n-1}q_n)(\phi_{n-1,1}y_1+\phi_{n-1,2}y_2+\cdots+\phi_{n-1,n-1}y_{n-1})\\&+\psi_{nn}q_n(\phi_{n1}y_1+\cdots+\phi_{nn}y_n)\\&-[\text{same expression with } \phi \text{ and } \psi \text{ interchanged}]\\&=\phi_{11}y_1q_1+(\phi_{21}y_1+\phi_{22}y_2)+\cdots+(\phi_{n1}y_1+\cdots+\phi_{nn}y_n)q_n.\end{aligned}\quad(9)$$

Equating coefficients of $y_\kappa q_\kappa$ gives

$$\phi_{\kappa\kappa}=0,\qquad(\kappa=1\ \ldots\ n),\quad(10)$$

and equating coefficients of $y_{\kappa-1}q_\kappa$,

$$\phi_{\kappa-1,\kappa-1}\psi_{\kappa,\kappa-1}+\phi_{\kappa,\kappa-1}\psi_{\kappa\kappa}-\phi_{\kappa,\kappa-1}\psi_{\kappa-1,\kappa-1}-\phi_{\kappa\kappa}\psi_{\kappa,\kappa-1}=\phi_{\kappa,\kappa-1},$$

which becomes, making use of (10),

$$\phi_{\kappa,\kappa-1}(\psi_{\kappa\kappa}-\psi_{\kappa-1,\kappa-1})=\phi_{\kappa,\kappa-1},$$

and therefore

$$\psi_{\kappa\kappa}-\psi_{\kappa-1,\kappa-1}=1\ \text{if}\ \phi_{\kappa,\kappa-1}\neq0.\quad(11)$$

Noticing that $\phi_{\kappa\kappa}$ and $\psi_{\kappa\kappa}$ $(\kappa = 1 \ldots\ldots n)$ are respectively the roots of the characteristic equations belonging to U_1 and U_2, (10) and (11) give the following theorem:

If U_1 and U_2 generate a two-parameter linear or linearoid group having the composition $(U_1 U_2) = U_1$, the characteristic equation belonging to U_1 will have all of its roots zero, while the roots of the characteristic equation belonging to U_2 will form an arithmetical progression with the common difference unity, provided that none of the coefficients of form $\phi_{\kappa,\kappa-1}$ are absent.

Unless all of the quantities $\phi_{\kappa,\kappa-1}$ are zero, there is at least one pair of roots differing by unity. If, on the contrary, these quantities are all zero, we obtain by equating coefficients of $y_{\kappa-2,\kappa}$,

$$\phi_{\kappa-2,\kappa-2}\psi_{\kappa,\kappa-2} + \phi_{\kappa-1,\kappa-2}\psi_{\kappa,\kappa-1} + \phi_{\kappa,\kappa-2}\psi_{\kappa\kappa} - \phi_{\kappa,\kappa-2}\psi_{\kappa-2,\kappa-2} \\ - \phi_{\kappa,\kappa-1}\psi_{\kappa-1,\kappa-2} - \phi_{\kappa\kappa}\psi_{\kappa,\kappa-2} = \phi_{\kappa,\kappa-2}.$$

According to the hypothesis, all the terms in the left member vanish except two. Therefore,

$$\phi_{\kappa,\kappa-2}(\psi_{\kappa\kappa} - \psi_{\kappa-2,\kappa-2}) = \phi_{\kappa,\kappa-2},$$

which proves, as before, that there is at least one pair of roots differing by unity unless all of the coefficients $\phi_{\kappa,\kappa-2}$ also vanish. Suppose more generally that all of the quantities $\phi_{\kappa,\kappa}, \phi_{\kappa,\kappa-1}, \phi_{\kappa,\kappa-2}, \ldots\ldots \phi_{\kappa,\kappa-s}$ vanish. Equating coefficients of $y_{\kappa-s-1}\, q_\kappa$ in (9), we obtain

$$\phi_{\kappa-s-1,\kappa-s-1}\psi_{\kappa,\kappa-s-1} + \phi_{\kappa-s,\kappa-s-1}\psi_{\kappa,\kappa-s} + \phi_{\kappa-s+1,\kappa-s-1}\psi_{\kappa,\kappa-s+1} + \cdots\cdots \\ + \phi_{\kappa-1,\kappa-s-1}\psi_{\kappa,\kappa-1} + \phi_{\kappa,\kappa-s-1}\psi_{\kappa\kappa} \\ - \phi_{\kappa,\kappa-s-1}\psi_{\kappa-s-1,\kappa-s-1} - \phi_{\kappa,\kappa-s}\psi_{\kappa-s,\kappa-s-1} - \phi_{\kappa,\kappa-s+1}\psi_{\kappa-s+1,\kappa-s-1} - \cdots\cdots \\ - \phi_{\kappa,\kappa-1}\psi_{\kappa-1,\kappa-s-1} - \phi_{\kappa\kappa}\psi_{\kappa,\kappa-s-1} = \phi_{\kappa,\kappa-s-1}. \tag{12}$$

Taking into account the assumption made, this reduces to

$$\phi_{\kappa,\kappa-s-1}(\psi_{\kappa\kappa} - \psi_{\kappa-s-1,\kappa-s-1}) = \phi_{\kappa,\kappa-s-1}. \tag{13}$$

Unless, therefore, every $\phi_{\kappa,\kappa-s}$ $(\kappa = 1 \ldots\ldots n;\ s = 0 \ldots\ldots \kappa - 1)$ vanishes, there will be at least one pair of roots differing by unity. All of the $\phi_{\kappa,\kappa-s}$'s cannot vanish, for this means that the infinitesimal transformation U_1 vanishes. Hence we conclude

The characteristic equation belonging to U_2 always has at least one pair of roots differing by unity.

The theorem that the characteristic equation belonging to $U_1(f)$ has only vanishing roots, is not to be confounded with a theorem proved by Killing which, on the surface, appears to be the same.* The characteristic equation employed by Killing is quite different from the one considered here.

The second part of the theorem can be read as a property of the discriminant of the characteristic equation belonging to U_2, namely: If U_1 and U_2 generate a two-parameter group of composition $(U_1 U_2) = a U_1$, the discriminant of the characteristic equation belonging to U_2 has in the general case the numerical value $\dfrac{[2 \,.\, 3 \,.\, 4 \,\ldots\ldots\, (n-1)]^{2n}}{[2^2\, 3^3\, 4^4 \,\ldots\ldots\, (n-1)^{n-1}]^2}\, a^{n(n-1)}$. More generally, all of the invariants of this equation are numerical, since they are functions of the differences of the roots.

§7.—*General Theorem.*

The theorem, proved in the last paragraph, can be immediately extended to r-parameter integrable groups. The infinitesimal transformations of an integrable linear or linearoid group can be chosen to satisfy the equation (Lie, "Continuierliche Gruppen," p. 537),

$$(U_i\, U_{i+\kappa}) = \sum_{1}^{i+\kappa-1} c_{i,\, i+\kappa,\, s}\, U_s, \qquad \begin{pmatrix} i = 1,\ 2 \,\ldots\ldots\, r \\ \varkappa = 1,\ 2 \,\ldots\ldots\, r - i \end{pmatrix}. \tag{1}$$

The infinitesimal transformations can be put into the form

$$U_\kappa = \phi_{11}^{(\kappa)} y_1 q_1 + (\phi_{21}^{(\kappa)} y_1 + \phi_{22}^{(\kappa)} y_2) q_2 + \ldots\ldots + (\phi_{n1}^{(\kappa)} y_1 + \ldots\ldots + \phi_{nn}^{(\kappa)} y_n) q_n. \tag{2}$$

If we assume

$$(U_\iota\, U_\kappa) = c_{i\kappa 1} U_1 + c_{i\kappa 2} U_2 + \ldots\ldots + c_{i\kappa r} U_r, \tag{3}$$

we obtain, by equating coefficients of $y_\lambda q_\lambda$, the system of equations

$$c_{i\kappa 1}\, \phi_{\lambda\lambda}^{(1)} + c_{i\kappa 2}\, \phi_{\lambda\lambda}^{(2)} + \ldots\ldots + c_{i\kappa r}\, \phi_{\lambda\lambda}^{(r)} = 0, \qquad (i,\ \varkappa,\ \lambda = 1 \,\ldots\ldots\, r). \tag{4}$$

Putting $i = \varkappa = 1$, equations (1) become

$$(U_1\, U_2) = c_{121}\, U_1.$$

For this case equations (4) become

$$c_{121}\, \phi_{\lambda\lambda}^{(1)} = 0, \qquad (\lambda = 1 \,\ldots\ldots\, r), \tag{5}$$

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* Lie, "Transformationsgruppen," Vol. III, p. 772.

hence, if $c_{121} \neq 0$, all of the roots of the characteristic equation belonging to U_1 are zero. We also have from (1)

$$\left.\begin{aligned}(U_1 U_3) &= c_{131} U_1 + c_{132} U_2, \\ (U_2 U_3) &= c_{231} U_1 + c_{232} U_2.\end{aligned}\right\}$$

Equations (4) become

$$\left.\begin{aligned}c_{131} \phi^{(1)}_{\lambda\lambda} + c_{132} \phi^{(2)}_{\lambda\lambda} &= 0, \\ c_{231} \phi^{(1)}_{\lambda\lambda} + c_{232} \phi^{(2)}_{\lambda\lambda} &= 0,\end{aligned}\right\}$$

from which it follows that $\phi^{(2)}_{\lambda\lambda} = 0$, if we assume the roots of U_1 all zero, and c_{132} and c_{232} not both zero. The general theorem is clear, and may be stated as follows:

If all of the roots of the characteristic equations belonging to the infinitesimal transformations $U_1 \ldots U_\kappa$ vanish, where these are the κ infinitesimal transformations of a κ-parameter invariant subgroup of an integrable group having the composition (1), *the same is true of the characteristic equation belonging to $U_{\kappa+1}$, provided that not all of the composition constants*

$$c_{1,\kappa+2,\kappa+1}, \quad c_{2,\kappa+2,\kappa+1}, \quad \ldots\ldots \quad c_{\kappa+1,\kappa+2,\kappa+1}$$

are equal to zero.

If these conditions are not fulfilled, we proceed as follows: Suppose the roots of the characteristic equation belonging to U_μ have been found by the method outlined above to be zero. Strike out the μ^{th} term in system (4) for each value of μ. If there is left any equation containing only a single term $c_{i\kappa\tau} \phi^{(\tau)}_{\lambda\lambda}$, we conclude that the roots of the characteristic equation belonging to U_τ are zero. We would then drop the τ^{th} term from the equations, and repeat the process until we obtained a system of equations each one of which consists of two or more terms. In this way we find all of the infinitesimal transformations whose characteristic equations have zero roots.

The second part of the theorem contained in the last paragraph can also be extended. It can be stated as follows:

If the characteristic equations belonging to the infinitesimal transformations of a p-parameter invariant subgroup of an r-parameter integrable linear or linearoid group have all their roots zero, and the roots of the characteristic equation belonging to U_{p+1} are not zero, then the differences between its roots are constants depending only upon the composition of the group, and no more than p of these differences are distinct, provided that not all the terms in $U_1 \ldots\ldots U_p$ of the form $\phi^{(\kappa)}_{\lambda,\lambda-1} y_{\lambda-1} q_\lambda$ $(\lambda = 1 \ldots\ldots n; \kappa = 1 \ldots\ldots p)$ vanish. If all of these terms for μ distinct values

of λ vanish, the theorem is still true for those differences which correspond to values of λ different from these. And in any case there will be at least one pair of roots differing by a constant.

The proof of this theorem is similar to that employed in the last paragraph for the special case there considered. Writing

$$\left.\begin{aligned} U_i &= \phi_{11}^{(i)} y_1 q_1 + (\phi_{21}^{(i)} y_1 + \phi_{22}^{(i)} y_2) q_2 + \dots + (\phi_{n1}^{(i)} y_1 + \dots + \phi_{nn}^{(i)} y_n) q_n, \\ U_{p+1} &= \psi_{11} y_1 q_1 + (\psi_{21} y_1 + \psi_{22} y_2) q_2 + \dots + (\psi_{n1} y_1 + \dots + \psi_{nn} y_n) q_n, \end{aligned}\right\} \quad (6)$$
$$(i = 1 \dots p),$$

with the conditions

$$\phi_{\lambda\lambda}^{(i)} = 0, \quad \psi_{\lambda\lambda} \neq 0, \qquad (i = 1 \dots p;\ \lambda = 1 \dots n), \tag{7}$$

equations (3) and (6) lead to the following:

$$\begin{aligned} (U_i U_{p+1}) = (\psi_{11} q_1 + \psi_{21} q_2 + \dots + \psi_{n1} q_n) \phi_{11}^{(i)} y_1 \\ + (\psi_{22} q_2 + \psi_{32} q_3 + \dots + \psi_{n2} q_n)(\phi_{21}^{(i)} y_1 + \phi_{22}^{(i)} y_2) \\ + \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ - [\text{same expression with } \phi \text{ and } \psi \text{ interchanged}] \\ = c_{i,p+1,1} U_1 + c_{i,p+1,2} U_2 + \dots + c_{i,p+1,p} U_p, \quad (8) \\ (i = 1 \dots p). \end{aligned}$$

Equating coefficients of $y_{\kappa-1} q_\kappa$, we get

$$\begin{aligned} \phi_{\kappa-1,\kappa-1}^{(i)} \psi_{\kappa\kappa} + \phi_{\kappa,\kappa-1}^{(i)} \psi_{\kappa\kappa} - \phi_{\kappa,\kappa-1}^{(i)} \psi_{\kappa-1,\kappa-1} - \phi_{\kappa\kappa}^{(i)} \psi_{\kappa,\kappa-1} \\ = c_{i,p+1,1} \phi_{\kappa,\kappa-1}^{(1)} + c_{i,p+2,2} \phi_{\kappa,\kappa-1}^{(2)} + \dots + c_{i,p+1,p} \phi_{\kappa,\kappa-1}^{(p)}. \quad (9) \end{aligned}$$

Equations (7) and (9) give the following system of equations:

$$\left.\begin{aligned} [c_{1,p+1,1} - (\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1})] \phi_{\kappa,\kappa-1}^{(1)} + c_{1,p+1,2} \phi_{\kappa,\kappa-1}^{(2)} + \dots \\ + c_{1,p+1,p} \phi_{\kappa,\kappa-1}^{(p)} = 0, \\ c_{2,p+1,1} \phi_{\kappa,\kappa-1}^{(1)} + [c_{2,p+1,2} - (\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1})] \phi_{\kappa,\kappa-1}^{(2)} + \dots \\ + c_{2,p+1,p} \phi_{\kappa,\kappa-1}^{(p)} = 0, \\ \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots \\ c_{p,p+1,1} \phi_{\kappa,\kappa-1}^{(1)} + c_{p,p+1,2} \phi_{\kappa,\kappa-1}^{(2)} + \dots \\ + [c_{p,p+1,p} - (\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1})] \phi_{\kappa,\kappa-1}^{(p)} = 0, \end{aligned}\right\} (10)$$
$$(\kappa = 2 \dots n).$$

Unless all of the ϕ's in (10) are zero, the determinant of the coefficients vanishes, leading to an equation of degree p for the determination of $\psi_{\kappa\kappa} - \psi_{\kappa-1,\kappa-1}$. Since $\psi_{\kappa\kappa}$ and $\psi_{\kappa-1,\kappa-1}$ are any consecutive roots, we conclude that the root differences are constants, and not more than p of these differences are distinct,

since they are all roots of the same equation of degree p whose coefficients moreover depend merely upon the composition of the group. This result is interesting with regard to the determinateness of the transformation U_{p+1}, if this is to be so determined as to generate with $U_1 \ldots . U_p$ a $p+1$-parameter group of given composition. Since the difference between two consecutive roots, considered as arranged in a certain order, must, according to the theorem, assume one of p constant values, which are fixed beforehand, it follows that after one root ρ_1 has been arbitrarily chosen, only a finite number of values are possible for the other roots.

As in the special case of the last paragraph, this theorem breaks down if the quantities $\phi^{\kappa}_{\lambda, \lambda-1}$ are all zero. Suppose, therefore, that all terms of the form $\phi^{(i)}_{\kappa, \kappa-s} y_{\kappa-s} q_{\kappa}$ $(s = 0 \ldots . s;\ \varkappa = 1 \ldots . n;\ i = 1 \ldots . p)$ vanish. Equating coefficients of $y_{\kappa-s-1} q_{\kappa}$, we get

$$\begin{aligned}
&\phi^{(i)}_{\kappa-s-1, \kappa-s-1} \psi_{\kappa, \kappa-s-1} + \phi^{(i)}_{\kappa-s, \kappa-s-1} \psi_{\kappa, \kappa-s} + \phi^{(i)}_{\kappa-s+1, \kappa-s-1} \psi_{\kappa, \kappa-s+1} + \ldots . \\
&\quad + \phi^{(i)}_{\kappa-1, \kappa-s-1} \psi_{\kappa, \kappa-1} + \phi^{(i)}_{\kappa, \kappa-s-1} \psi_{\kappa\kappa} \\
&\quad - \phi^{(i)}_{\kappa, \kappa-s-1} \psi_{\kappa-s-1, \kappa-s-1} - \phi^{(i)}_{\kappa, \kappa-s} \psi_{\kappa-s, \kappa-s-1} - \phi^{(i)}_{\kappa, \kappa-s+1} \psi_{\kappa-s+1, \kappa-s-1} - \ldots . \\
&\quad - \phi^{(i)}_{\kappa, \kappa-1} \psi_{\kappa-1, \kappa-s-1} - \phi^{(i)}_{\kappa\kappa} \psi_{\kappa, \kappa-s-1} = c_{1, p+1, 1} \phi^{(1)}_{\kappa-s-1} \\
&\quad + c_{1, p+1, 2} \phi^{(2)}_{\kappa, \kappa-s-1} + \ldots . + c_{1, p+1, p} \phi^{(p)}_{\kappa, \kappa-s-1}.
\end{aligned} \tag{11}$$

All except the two middle terms in the left member of this equation vanish. The following system is therefore obtained :

$$\left.\begin{aligned}
&[c_{1, p+1, 1} - (\psi_{\kappa\kappa} - \psi_{\kappa-s-1, \kappa-s-1})]\, \phi^{(1)}_{\kappa, \kappa-s-1} + c_{1, p+1, 2} \phi^{(2)}_{\kappa, \kappa-s-1} + \ldots . \\
&\qquad\qquad + c_{1, p+1, p} \phi^{(p)}_{\kappa, \kappa-s-1} = 0, \\
&c_{2, p+1, 1} \phi^{(1)}_{\kappa, \kappa-s-1} + [c_{2, p+1, 2} - (\psi_{\kappa\kappa} - \psi_{\kappa-s-1, \kappa-s-1})]\, \phi^{(2)}_{\kappa, \kappa-s-1} + \ldots . \\
&\qquad\qquad + c_{2, p+1, p} \phi^{(p)}_{\kappa, \kappa-s-1} = 0, \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&c_{p, p+1, 1} \phi^{(1)}_{\kappa, \kappa-s-1} + c_{p, p+1, 2} \phi^{(2)}_{\kappa, \kappa-s-1} + \ldots . \\
&\qquad\qquad + [c_{p, p+1, p} - (\psi_{\kappa\kappa} - \psi_{\kappa-s-1, \kappa-s-1})]\, \phi^{(p)}_{\kappa, \kappa-s-1} = 0.
\end{aligned}\right\} \tag{12}$$

This system in $\phi_{\kappa, \kappa-s-1}$ is identical with (10). Unless, therefore, all of the quantities $\phi_{\kappa, \kappa-s-1}$ are zero, we conclude, as before, that the differences $\psi_{\kappa\kappa} - \psi_{\kappa-s-1, \kappa-s-1}$ are the roots of an equation of degree p, whose coefficients are functions of the composition constants. Since s may assume all values from 1 to $\mu - 2$, we conclude that there is at least one pair of roots of the characteristic equation belonging to U_{p+1} whose difference is a constant, otherwise all of the infinitesimal transformations of the p-parameter subgroup would vanish identically.

From the results obtained above follows the additional theorem: *If $U_1 \ldots . U_r$ generate an r-parameter integrable linear or linearoid group which have been put in the form demanded by* (1), *the characteristic equations belonging to $U_1 \ldots . U_{r-1}$ have all their roots equal to zero, provided that for each value of $\varkappa$ from* 1 *to r not all the composition constants $c_{i,\,\kappa,\,\kappa-1}$* ($i = 1 \ldots . \varkappa - 1$) *vanish. Under these same conditions the differences between the roots of the characteristic equation belonging to U_r are constants, no more than r of these differences being distinct, provided not all the coefficients in $U_1 \ldots . U_{r-1}$ of the form $\phi^{(\kappa)}_{\lambda,\,\lambda-1}$* ($\lambda = 1 \ldots . n$; $\varkappa = 1 \ldots . r - 1$) *vanish. And in any case, provided the first condition holds, there is always one pair of roots whose difference is a constant.*

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